University of Dayton

Homework assignment 4
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AEE 401-Aerodynamics 1
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Anderson's Problems

3.8. (5)

Given:

Uniform flow with velocity V_∞

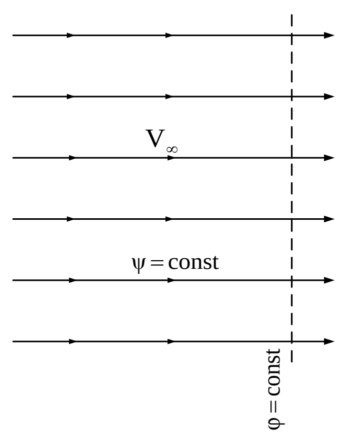

Figure 1: schematic diagram for problem 3.8.

$$\boxed{u = V_\infty = const} \tag{1}$$

Find:

Show that this flow is a physically possible incompressible flow and that it is irrotational.

Solution:

The partial differential equation of the conservation of mass as given in equation (2.52) is a as follows;

$$\boxed{\frac{\partial\rho}{\partial t}+\nabla.(\rho V)=0} \tag{2.52}$$

The first term of equation, which is the time rate of change in volume, (2.52) will diminish if the flow is incompressible. Hence,

$$\boxed{\nabla.V=0 \Longrightarrow \nabla.V=0=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}} \quad (1) \tag{2}$$

$$\frac{\partial u}{\partial x}=0, \qquad \frac{\partial v}{\partial y}=0, \qquad \frac{\partial w}{\partial z}=0 \tag{3}$$

Hence,

$$\boxed{\nabla.V=0 \Longrightarrow The\ flow\ is\ incompressible} \quad (1) \tag{4}$$

We defined in chapter 2 that the vorticity is the curl of the velocity vector as follows;

$$\boxed{\xi=\nabla\times\vec{V}} \tag{2.129}$$

Hence, in Cartesian coordinates;

$$\boxed{\xi=\nabla\times\vec{V}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{u} & v & \mathrm{w} \end{vmatrix}} \quad (1) \tag{5}$$

$$\xi=\nabla\times\vec{V}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{u} & v & \mathrm{w} \end{vmatrix}=i(0)-j(0)+k(0)=0 \quad (1) \tag{6}$$

Since the vorticity is zero, this implies that the flow is irrational. (1)

3.9. (15)

Given:

Source flow

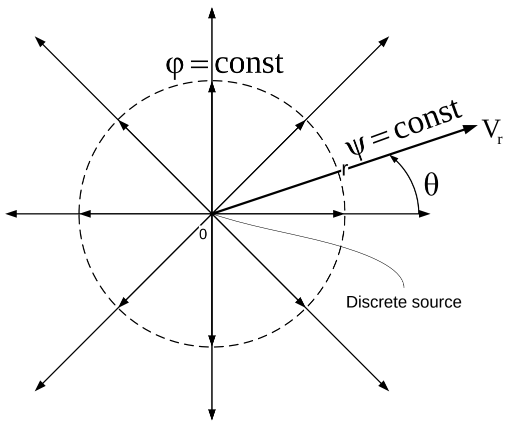

Figure 2: source flow.

$$\boxed{V_r = \frac{Q}{2\pi r}} \quad (1)$$

Find:

Show that a source flow is a physically possible incompressible flow everywhere except at the origin. Also, show that it is irrotational everywhere.

Solution:

The partial differential equation of the conservation of mass as given in equation (2.52) is a as follows;

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0} \quad (2.52)$$

The first term of equation, which is the time rate of change in volume, (2.52) will diminish if the flow is incompressible. Hence,

$$\boxed{\nabla.V = 0 \Longrightarrow \nabla.V = 0 = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}} \quad 1 \quad (2)$$

Hence,

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{Q}{2\pi r}\right) = 0, \qquad \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = 0, \qquad \frac{\partial V_z}{\partial z} = 0 \quad (3)$$

$$\boxed{\nabla . V = 0 \Longrightarrow The\ flow\ is\ \ incompressible} \qquad ① \tag{4}$$

Hence, the flow is a physically possible incompressible flow everywhere except at the origin. Now, our aim is to prove that at the origin the flow will be compressible. To do so, let's consider a cylinder given in figure 2 with a radius r. Also, let's consider a unit depth. The volume flow across this cylindrical surface is;

$$\oiint_S \vec{V} . d\vec{S} \qquad ① \tag{5}$$

We have to be aware that the volume flow Q for a source flow introduced in equation (1) is per unit depth. This leads us to the following relation;

$$\oiint_S \vec{V} . d\vec{S} = Q = const \qquad ② \tag{6}$$

From the divergence theorem, which transfers the surface integral to a volume integral or vise versa, can be given in the following relation;

$$\oiint_S \vec{V} . d\vec{S} = \oiiint_V (\nabla . \vec{V}) dV \qquad ② \tag{7}$$

Combining equation (6) and (7) leads to;

$$\oiiint_V (\nabla . \vec{V}) dV = Q = const \qquad ① \tag{8}$$

From equation (8), we can conclude that as r approaches 0, the volume under the cylinder will be an infinitesimal area, leading to;

$$(\nabla . \vec{V}) dV = Q \ \Longrightarrow \boxed{(\nabla . \vec{V}) = \frac{Q}{dV}} \qquad ② \tag{9}$$

Taking the limit of equation (9);

$$(\nabla . \vec{V}) = \lim_{dV \to 0} \frac{Q}{dV} \Longrightarrow \nabla . \vec{V} = \infty \qquad ② \tag{10}$$

We see as the dV approaches zero, the values of $\nabla.\vec{V}$ will approach the infinity, leading to the conclusion that the divergence of the velocity is finite. Since the divergence of the velocity is a finite values, this means that the flow is compressible at the origin of a source flow.

The last part is to test whether the source flow is rotational or irrotational. The vorticity in polar coordinates is given in the following relation;

$$\xi = \nabla \times \vec{V} = \frac{1}{r}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \mathrm{V}_r & r\mathrm{V}_\theta & \mathrm{V}_z \end{vmatrix} \quad (11)$$

2

Substituting equations (1) into (11) results;

$$\xi = \nabla \times \vec{V} = \frac{1}{r}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{Q}{2\pi r} & 0 & 0 \end{vmatrix} = i(0) - j\left(0 - \frac{\partial}{\partial z}\left(\frac{Q}{2\pi r}\right)\right) + k\left(\frac{\partial}{\partial \theta}\left(\frac{Q}{2\pi r}\right)\right) = 0 \quad (12)$$

Since the vorticity is zero, this implies that the flow is irrotational everywhere. 1

3.10. (5)

Given:

Velocity potential and stream function.

$$\boxed{\phi = V_\infty x} \tag{3.53}$$

$$\boxed{\psi = V_\infty y} \tag{3.55}$$

Find:

Prove that the velocity potential and stream function for a uniform flow given in equation (3.53) and (3.55) satisfy Laplace's equation.

Solution:

Recall Laplace's equations (3.40) and (3.46);

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 = \nabla^2 \phi} \tag{3.40}$$

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 = \nabla^2 \psi} \tag{3.46}$$

Substituting equation (3.53) into (3.40) yields;

$$\frac{\partial^2}{\partial x^2}(V_\infty x) + \frac{\partial^2}{\partial y^2}(V_\infty x) + \frac{\partial^2}{\partial z^2}(V_\infty x) = 0 \Longrightarrow \boxed{\nabla^2 \phi = 0} \tag{1}$$

2.5

Since $\nabla^2 \phi = 0$, means that the velocity potential satisfies Laplace's equation. Similarly, for stream function;

$$\frac{\partial^2}{\partial x^2}(V_\infty y) + \frac{\partial^2}{\partial y^2}(V_\infty y) + \frac{\partial^2}{\partial z^2}(V_\infty y) = 0 \Longrightarrow \boxed{\nabla^2 \psi = 0} \tag{2}$$

2.5

Since $\nabla^2 \psi = 0$, means that the stream function satisfies Laplace's equation.

3.11. (5)

Given:

Velocity potential and stream function.

$$\boxed{\phi = \frac{Q}{2\pi}\ln r} \tag{3.67}$$

$$\boxed{\psi = \frac{Q}{2\pi}\theta} \tag{3.72}$$

Find:

Prove that the velocity potential and stream function for a source flow given in equation (3.67) and (3.72) satisfy Laplace's equation.

Solution:

Recall Laplace's equations (3.42);

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 = \nabla^2 \phi} \tag{3.42}$$

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0 = \nabla^2 \psi} \tag{3.46}$$

Substituting equation (3.67) into (3.40) yields;

2.5

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{Q}{2\pi}\ln r\right)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{Q}{2\pi}\ln r\right) + \frac{\partial^2}{\partial z^2}\left(\frac{Q}{2\pi}\ln r\right) = 0 \Longrightarrow \boxed{\nabla^2 \phi = 0} \tag{1}$$

Since $\nabla^2 \phi = 0$, means that the velocity potential satisfies Laplace's equation. Similarly, for stream function;

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{Q}{2\pi}\theta\right)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{Q}{2\pi}\theta\right) = 0 \Longrightarrow \boxed{\nabla^2 \psi = 0} \tag{2}$$

2.5

Since $\nabla^2 \psi = 0$, means that the stream function satisfies Laplace's equation.

3.15. (15)

Given:

Non lifting flow over a circular cylinder.

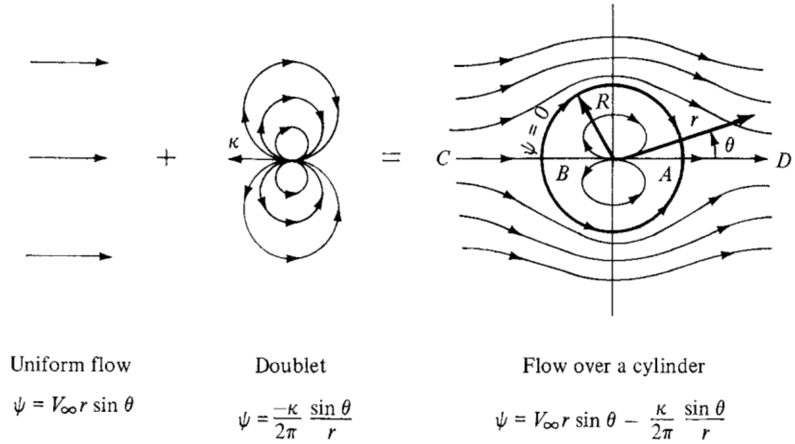

Figure 2: non-lifting flow over a cylinder.

Find:

Derive an expression for the pressure coefficient at an arbitrary point (r, θ) in this flow, and show that it reduces to equation (3.101) on the surface of the cylinder.

Solution:

The stream function for the combined flow for the non-lifting flow over a circular cylinder is given as follows;

$$\psi = V_\infty r \sin\theta \left[1 - \frac{R^2}{r^2}\right] \tag{3.91}$$

Where the radial and tangential components are given in the following relations;

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = V_\infty \cos\theta \left[1 - \frac{R^2}{r^2}\right] \tag{3.93}$$

2

$$V_\theta = -\frac{\partial\psi}{\partial r} = -V_\infty \sin\theta \left[1 + \frac{R^2}{r^2}\right] \tag{3.94}$$

2

From Pythagoras, the magnitude can be found as follows;

$$V^2 = V_r^2 + V_\theta^2 \tag{1}$$

2

Hence,

$$V^2 = \left(V_\infty \cos\theta \left[1 - \frac{R^2}{r^2}\right]\right)^2 + \left(-V_\infty \sin\theta \left[1 + \frac{R^2}{r^2}\right]\right)^2 \tag{2}$$

$$V^2 = \left(\left[V_\infty \cos\theta - V_\infty \cos\theta \frac{R^2}{r^2}\right]\right)^2 + \left(\left[-V_\infty \sin\theta - V_\infty \sin\theta \frac{R^2}{r^2}\right]\right)^2 \tag{3}$$

3

$$\begin{aligned} V^2 = & \left[V_\infty^2 \cos^2\theta - 2V_\infty^2 \cos^2\theta \frac{R^2}{r^2} + V_\infty^2 \cos^2\theta \frac{R^4}{r^4}\right] \\ & + \left[V_\infty^2 \sin^2\theta + 2V_\infty^2 \sin^2\theta \frac{R^2}{r^2} + V_\infty^2 \sin^2\theta \frac{R^4}{r^4}\right] \end{aligned} \tag{4}$$

At the surface, which corresponds to $r = R$, we get;

$$\begin{aligned} V^2 = & [V_\infty^2 \cos^2\theta - 2V_\infty^2 \cos^2\theta + V_\infty^2 \cos^2\theta] \\ & + [V_\infty^2 \sin^2\theta + 2V_\infty^2 \sin^2\theta + V_\infty^2 \sin^2\theta] \end{aligned} \tag{5}$$

$$V^2 = [V_\infty^2 \sin^2\theta + 2V_\infty^2 \sin^2\theta + V_\infty^2 \sin^2\theta] = 4V_\infty^2 \sin^2\theta \tag{6}$$

2

Recall equation (3.38),

$$\boxed{C_p = 1 - \frac{V^2}{V_\infty^2}} \tag{3.38}$$

Substituting equation (6) into (3.38) results;

$$\boxed{C_p = 1 - \frac{4V_\infty^2 \sin^2\theta}{V_\infty^2}} \tag{3.38}$$

2

$$\boxed{C_p = 1 - 4\sin^2\theta} \tag{3.38}$$

2

Collicot's Problems

3.3. (15)

Given:

Source and sink is as shown in figure 3.

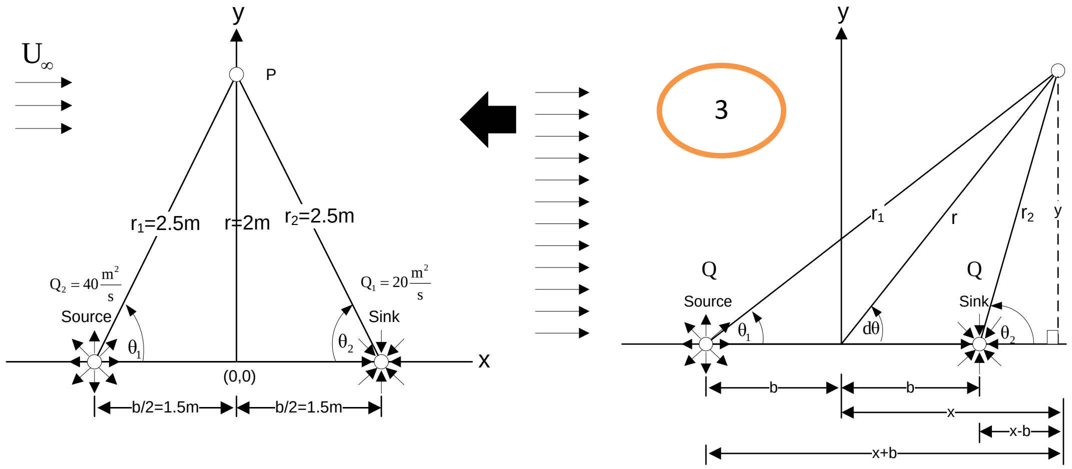

Figure 3: schematic diagram of problem 3.3 Collicot.

Find:

Find the velocity at this point and the velocity of the undistributed stream.

Solution:

Figure (3) shows a schematic diagram of the problem and assuming that b is the distance of 3 m between the source and the sink. Furthermore, assuming that the source is located at $(-\,b/2\,,0)$ and the sink at $(b/2\,,0)$.

Denote the equidistant point to be P, and it is located at $(0,r)$, and let $r_1 = r_2 = 2.5m$ be the distance from P both the source and the sink. We are going to assume that the subscript 1 denotes to the sink and the subscript 2 denotes to the source. Also, knowing that $Q_2 = 2Q_1$. Applying Pythagoras theorem leads to;

$$\boxed{r_2^2 = r^2 + \left(\frac{b}{2}\right)^2} \qquad (1)$$

Hence,

$$r^2 = r_2^2 - \left(\frac{b}{2}\right)^2 = (2.5)^2 - \left(\frac{3}{2}\right)^2 \Rightarrow \boxed{r = 2m} \quad \text{1} \quad (2)$$

Also, from figure (3), $\theta_1 and\ \theta_2$ can be found as follows;

$$\boxed{\theta_2 = \tan^{-1}\left(\frac{r}{x - 1.5}\right), \qquad \theta_1 = \tan^{-1}\left(\frac{r}{x + 1.5}\right)} \quad \text{2} \quad (3)$$

Recalling equations (3.53), (3.55), (3.67), and (3.72);

Uniform flow:

$$\boxed{\phi = U_\infty x = U_\infty r \cos\theta} \quad (3.53)$$

$$\boxed{\psi = U_\infty y = U_\infty r \sin\theta} \quad (3.55)$$

Source flow:

$$\boxed{\phi = \frac{Q}{2\pi} \ln r} \quad (3.67)$$

$$\boxed{\psi = \frac{Q}{2\pi}\theta} \quad (3.72)$$

Velocity Components:

$$\boxed{u = \frac{\partial\psi}{\partial y}} \quad (4)$$

$$\boxed{v = -\frac{\partial\psi}{\partial x}} \quad (5)$$

Hence, the total stream function of the three flows are;

$$\boxed{\psi = \psi_{source} + \psi_{sink} + \psi_{unifrom}} \quad \text{1} \quad (6)$$

$$\boxed{\psi = \frac{Q_2}{2\pi}\theta_2 - \frac{Q_1}{2\pi}\theta_1 + U_\infty y} \quad \text{1} \quad (7)$$

Substituting equation (3) into (7) results;

$$\psi = \frac{Q_2}{2\pi}\tan^{-1}\left(\frac{y}{x-1.5}\right) - \frac{Q_1}{2\pi}\tan^{-1}\left(\frac{y}{x+1.5}\right) + U_\infty y \quad (8)$$

$$\boxed{\psi = \frac{Q_1}{2\pi}\left[\tan^{-1}\left(\frac{r}{x-1.5}\right) - 2\tan^{-1}\left(\frac{r}{x+1.5}\right)\right] + U_\infty y} \quad 2 \quad (9)$$

Deriving the stream function of the total flow of equation (9) with respect to x yields;

$$\frac{\partial\psi}{\partial x} = v = -\frac{Q_1}{2\pi}\left[\left(\frac{-r}{(x-1.5)^2+r^2}\right) - 2\left(\frac{-r}{(x+1.5)^2+r^2}\right)\right] + 0 \quad (10)$$

At the equidistant point P, we have $(x, y) = (0,2)$, hence;

$$\frac{\partial\psi}{\partial x} = v = -\frac{Q_1}{2\pi}\left[\left(\frac{-2}{(0-1.5)^2+4}\right) - 2\left(\frac{-2}{(0+1.5)^2+4}\right)\right] \quad 1 \quad (11)$$

Hence, the velocity at the point is;

$$\boxed{v = 1.0186\, m/s} \quad 1 \quad (12)$$

To find the velocity of the undistributed stream, derive the stream function given in (9) with respect to y. hence,

$$\frac{\partial\psi}{\partial y} = u = \frac{Q_1}{2\pi}\left[\left(\frac{x-1.5}{(x-1.5)^2+r^2}\right) - 2\left(\frac{x+1.5}{(x+1.5)^2+r^2}\right)\right] + U_\infty \quad (13)$$

Again, at the equidistant point P, we have $(x, y) = (0,2)$, hence;

$$0 = \frac{Q_1}{2\pi}\left[\left(\frac{-1.5}{(0-1.5)^2+4}\right) - 2\left(\frac{1.5}{(0+1.5)^2+4}\right)\right] + U_\infty \quad 1 \quad (14)$$

Hence, the velocity of the undistributed stream is;

$$\boxed{U_\infty = 2.292\, m/s} \quad 1 \quad (15)$$

3.6.(10)

Given:

Two dimensional source of strength m.

Find:

1. Determine the stream function for this flow.
2. Sketch the resultant filed of flow due to three such sources, each of strength m, located at the vertices of an equilateral triangle. (U of L).

Solution:

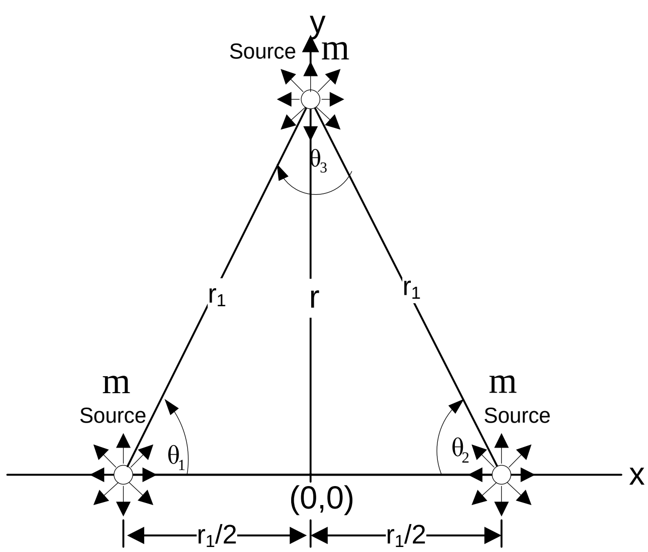

Figure 4: schematic diagram of problem 3.6 Collicot.

From figure (4), $\theta_1,\ \theta_2, and\ \theta_3$ can be found as follows;

1

$$\theta_3 = \tan^{-1}\left(\frac{r}{x}\right), \qquad \theta_2 = \tan^{-1}\left(\frac{r}{x + r_1/2}\right), \qquad \theta_1 = \tan^{-1}\left(\frac{r}{x - r_1/2}\right) \tag{1}$$

Hence, the total stream function of the three flows are;

$$\psi = \psi_{source} + \psi_{source} + \psi_{source} \tag{2}$$

2

$$\psi = \frac{m}{2\pi}[\theta_3 + \theta_2 + \theta_1] \tag{3}$$

Substituting equation (1) into (3) results;

2

$$\psi = \frac{Q_1}{2\pi}\left[\tan^{-1}\left(\frac{r}{x - r_1/2}\right) + \tan^{-1}\left(\frac{r}{x + r_1/2}\right) + \tan^{-1}\left(\frac{r}{x}\right)\right] \tag{4}$$

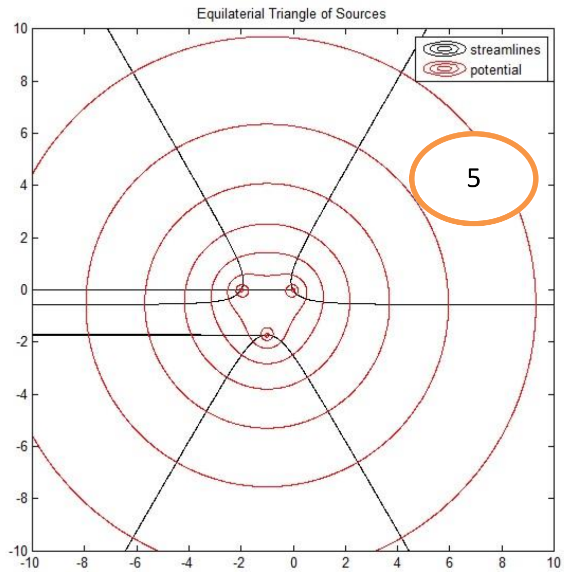

Figure 5: plot for problem 3.6 Collicot.

3.9. (15)

Given:

A line source is immersed in a uniform stream as shown in figure 5.

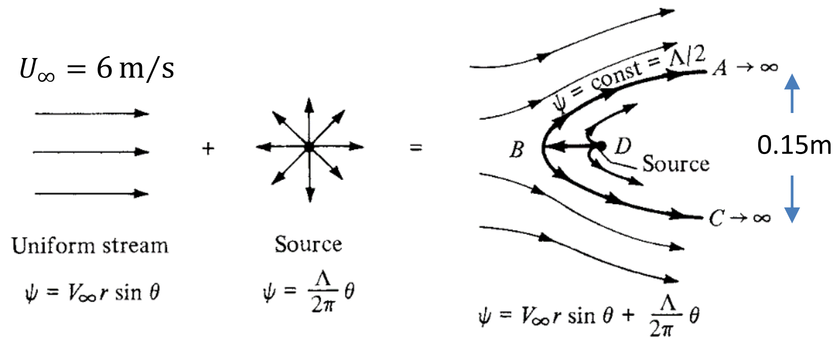

Figure 5: non-lifting flow over a semi-infinite body.

$$U_\infty = 6\,\mathrm{m/s}, \qquad t = 0.15m$$

Find:

Show that the resultant flow, if irrotational, may represent the flow past a 2-d fairing. If the maximum thickness of the fairing is 0.15m and the undisturbed velocity of the stream is $6\,\mathrm{m/s}$, determine the strength and location of the source. Also, obtain an expression for the pressure at any point on the surface of the fairing, taking the pressure at infinity as a datum.

1. Q?
2. The stagnation point location.
3. An expression the pressure at any point on the surface of the fairing.

Note: $\Lambda = Q$

Solution:

Recalling equations (3.53), (3.55), (3.67), and (3.72);

Uniform flow:

$$\boxed{\phi = U_\infty x = U_\infty r\cos\theta} \qquad (3.53)$$

$$\boxed{\psi = U_\infty y = U_\infty r\sin\theta} \qquad (3.55)$$

Source flow:

$$\boxed{\phi = \frac{Q}{2\pi}\ln r} \quad (3.67)$$

$$\boxed{\psi = \frac{Q}{2\pi}\theta} \quad (3.72)$$

Velocity Components:

$$\boxed{V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}} \quad (1)$$

$$\boxed{V_\theta = -\frac{\partial \psi}{\partial r}} \quad (2)$$

Hence, the total stream function of the two flows are;

1

$$\boxed{\psi = \psi_{source} + \psi_{unifrom}} \quad (3)$$

Substituting equation (3.55) and (3.72) into (3) gives;

2

$$\boxed{\psi = \frac{Q}{2\pi}\theta + U_\infty y = \frac{Q}{2\pi}\theta + U_\infty r \sin\theta} \quad (4)$$

Applying equations (1) and (2) into (4) to obtain the velocity components, hence,

$$\boxed{V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{Q}{2\pi r} + U_\infty \cos\theta} \quad (5)$$

$$\boxed{V_\theta = -\frac{\partial \psi}{\partial r} = -U_\infty \sin\theta} \quad (6)$$

$$\boxed{V^2 = V_r^2 + V_\infty^2 = \left(\frac{Q}{2\pi r} + U_\infty \cos\theta\right)^2 + (-U_\infty \sin\theta)^2} \quad (7)$$

1. As we see from figure (5), the maximum thickness of the semi-infinite body happens when $\psi = const = Q/2$. Therefore, when x approaches infinity, the values of y approaches 0.15. From equation (4), we can get the following;

$$\lim_{x\to\infty}\left(\frac{Q}{2}\right) = \lim_{x\to\infty}\left(\frac{Q}{2\pi}\tan^{-1}\left(\frac{0.15/2}{\infty}\right) + \frac{0.15}{2}U_\infty\right) \quad (8)$$

2 $$\frac{Q}{2} = \frac{0.15}{2} U_\infty \Longrightarrow \boxed{Q = 0.9\, m^2/s} \qquad (9)$$

2. At the stagnation points, which is showing figure 5 point B, the velocity components in equations (5) and (6) will be zero, leading to $(r, \theta) = \left(\frac{Q}{2\pi U_\infty}, \pi\right)$, hence,

5 $$0 = \frac{Q}{2\pi r} - U_\infty = \frac{0.9}{2\pi r} - 6 \Longrightarrow \boxed{r = 0.0287m} \qquad (9)$$

3. Recall (3.36) and (3.38);

$$\boxed{C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2}} \qquad (3.36)$$

$$\boxed{C_p = 1 - \frac{V^2}{U_\infty^2}} \qquad (3.38)$$

The streamline parallel to the surface of the semi-infinite half body is given by

1 $$\boxed{\frac{Q}{2} = \frac{Q}{2\pi}\theta + U_\infty r \sin\theta} \qquad (10)$$

Solving equation (10) for r yields;

1 $$\boxed{r = \frac{Q(\pi - \theta)}{2\pi U_\infty \sin\theta}} \qquad (11)$$

Substituting equation (7) into (3.38) gives;

1 $$C_p = 1 - \frac{\left(\frac{Q}{2\pi r} + U_\infty \cos\theta\right)^2 + (-U_\infty \sin\theta)^2}{U_\infty^2} \qquad (12)$$

$$C_p = 1 - \frac{\frac{Q^2}{4\pi^2 r^2} + \frac{Q}{\pi r} U_\infty \cos\theta + \overbrace{U_\infty^2 \cos^2\theta + U_\infty^2 \sin^2\theta}^{\cos^2\theta + \sin^2\theta = 1}}{U_\infty^2} \qquad (13)$$

$$C_p = 1 - \frac{\frac{Q^2 4\pi^2 U_\infty^2 \sin^2\theta}{4\pi^2 Q^2 (\pi - \theta)^2} + \frac{Q 2\pi U_\infty \sin\theta}{\pi Q(\pi - \theta)} U_\infty \cos\theta + U_\infty^2}{U_\infty^2} \qquad (14)$$

$$C_p = 1 - \frac{\frac{U_\infty^2 \sin^2\theta}{(\pi-\theta)^2} + \frac{2U_\infty^2 \sin\theta\cos\theta}{(\pi-\theta)} + U_\infty^2}{U_\infty^2} \tag{15}$$

1

$$C_p = 1 - \left[\frac{\sin^2\theta}{(\pi-\theta)^2} + \frac{2\sin\theta\cos\theta}{(\pi-\theta)} + 1\right] \tag{16}$$

Substituting equation (16) into (3.36) gives;

$$\frac{p-p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - \left[\frac{\sin^2\theta}{(\pi-\theta)^2} + \frac{2\sin\theta\cos\theta}{(\pi-\theta)} + 1\right] \tag{17}$$

1

$$\boxed{p-p_\infty = \frac{1}{2}\rho U_\infty^2\left(1 - \left[\frac{\sin^2\theta}{(\pi-\theta)^2} + \frac{2\sin\theta\cos\theta}{(\pi-\theta)} + 1\right]\right)} \tag{18}$$

3.12. (15)

Given:

Find:

1. State the stream function and velocity potential for each of the motions induced by a source, vortex, and doublet in a two-dimensional incompressible fluid.
2. Show that a doublet may be regarded as either of the following:

a) The limiting case of a source and sink.

b) The limiting case of equal and opposite vortices, clearly indicating the direction of the resultant doublet. (U of L)

Solution:

1.

3

	Stream Function	Velocity Potential
Source	$\psi = \frac{Q}{2\pi}\theta$	$\phi = \frac{Q}{2\pi}\ln r$
Sink	$\psi = -\frac{Q}{2\pi}\theta$	$\phi = -\frac{Q}{2\pi}\ln r$
Vortex	$\psi = \frac{\Gamma}{2\pi}\ln r + C$	$\phi = \frac{-\Gamma}{2\pi}\theta$
Doublet	$\psi = \frac{-\kappa}{2\pi}\frac{\sin\theta}{r}$	$\phi = \frac{\kappa}{2\pi}\frac{\cos\theta}{r}$

2. a).

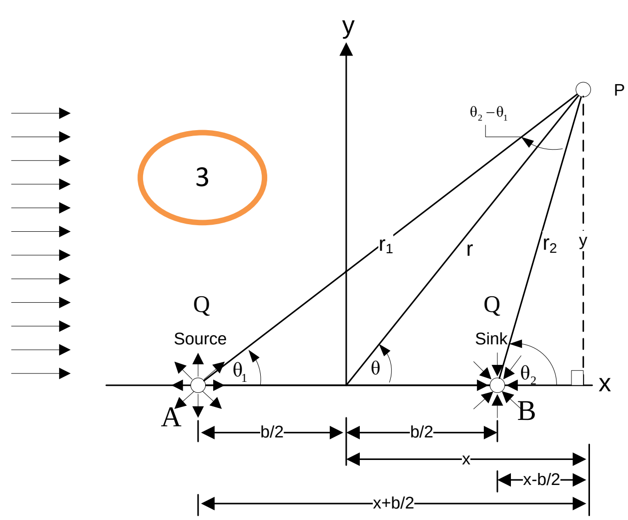

Figure 6: schematic diagram of problem 3.12a Collicot.

From figure (6), as b approaches zero, both θ_1 and θ_2 will approach θ; and both PA and PB will approach r. **<u>By the sine rule,</u>**

1

$$\frac{\sin(\theta_2 - \theta_1)}{b} = \frac{\sin\theta_2}{PA} \qquad (1)$$

Hence,

1

$$\boxed{\sin(\theta_2 - \theta_1) = \sin\theta_2 \times \frac{b}{PA}} \qquad (2)$$

So as b approaches zero;

$$\boxed{\sin(\theta_2 - \theta_1) \to \sin\theta \frac{b}{PA} = \frac{y}{r} \times \frac{b}{r}} \qquad (3)$$

Since a small angle (in radians) is equal to its sine, this can be written;

1

$$\boxed{(\theta_2 - \theta_1) = \frac{by}{r^2}} \qquad (4)$$

Now, the stream function for source and sink is given by

1

$$\boxed{\psi = \psi_{source} + \psi_{sink}} \qquad (5)$$

$$\psi = \frac{\mathrm{Q}}{2\pi}\theta_1 - \frac{\mathrm{Q}}{2\pi}\theta_2 = \frac{\mathrm{Q}}{2\pi}(\theta_1 - \theta_2) \Longrightarrow \boxed{\psi = \frac{\mathrm{Q}}{2\pi}\frac{-by}{r^2}} \qquad (6)$$

However, $b \times Q = k$, hence,

1

$$\boxed{\psi = \left(\frac{-\mathrm{k}}{2\pi}\right)\frac{y}{r^2} = \left(\frac{-\mathrm{k}}{2\pi}\right)\frac{r\sin(\theta)}{x^2 + y^2} = \left(\frac{-\mathrm{k}}{2\pi}\right)\frac{\sin(\theta)}{r}} \qquad (7)$$

2. b). To keep a very thing consistent, let's use the velocity potential keeping the same result we obtained for θ_1, and θ_2. Hence,

1

$$\boxed{\phi = \phi_{vortex} + \phi_{voetex}} \qquad (8)$$

1

$$\phi = \frac{\Gamma}{2\pi}\theta_1 - \frac{\Gamma}{2\pi}\theta_2 = \phi = \frac{\Gamma}{2\pi}(\theta_1 - \theta_2) \qquad (9)$$

Substituting equation (4) into (9) yields;

1

$$\phi = \frac{\Gamma}{2\pi}\left(\frac{bx}{r^2}\right) \tag{10}$$

However, $\Gamma \times Q = k$, hence,

1

$$\phi = \frac{k}{2\pi}\frac{\cos\theta}{r} \tag{11}$$

We will graphically show that a pair of vortices and a source/sink pair appropriately placed very close together approaches the flow field of a doublet. We will do this by adding a uniform stream to each of the pair of singularities and illustrate that the flow field obtained approaches a flow around a circle. Since this is what is obtained when a doublet is placed in a uniform stream, the questions raised in the problem statement are answered in Fig. 3. The MATLAB script applied to produce Fig. 3 is given in Table 3. It is worth noting that the strengths of the sources and the strengths of the vortices need to be increased substantially from unity to produce a circle of radius of order 1. (To show mathematically in the limit of the separation distance that the two pairs of singular solutions approach a doublet as their strengths approach infinity in such a way that the strength of the vortex doublet or source doublet approaches the finite doublet strength in the formula for the doublet. This mathematical demonstration is beyond the scope of this course.)